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Benjamin-Feir-type instability in a saturated ferrite: Transition between focusing and defocusing regimes for polarized electromagnetic waves

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We prove the existence of an oscillatory instability of the Benjamin-Feir type for electromagnetic propagation in a saturated ferrite. We do this by reducing the nonlinear equations governing the propagation of electromagnetic waves in such a medium to the nonlinear Schrödinger equation. We characterize regions where focusing or defocusing of the initial carrier envelope occurs in a function of three physical parameters: the phase velocity, the quotient between the external magnetic field and the magnetization of saturation, and a third one related to the angle between the direction of propagation of the carrier wave and the external magnetic field. We show that there exist points of transition between focusing and defocusing regimes for left elliptically polarized waves. No such point exists for right elliptically polarized waves. We show that all circularly polarized waves propagating parallel to the external magnetic field are stable (unstable) if they have negative (positive) helicity.

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I. INTRODUCTION

The study of electromagnetic wave propagation in ferro- or ferrimagnetic media is not only of interest theoretically but also practically, particularly in connection with the behavior of ferrite devices at microwave frequencies such as ferrite-loaded waveguides [1]. The propagation of electromagnetic waves in a ferromagnet obeys nonlinear equations with dispersion and dissipation. The linear theory has been investigated extensively by many authors. In particular, in Ref. [2] this approach provided a good explanation for phenomena such as cutoffs and resonances.

Results in the nonlinear theory are also known, but they are always partial and often based on drastic approximations of the initial dynamical equations. Recently, Nakata [3] began a rigorous study of the nonlinear case, investigating nonlinear propagation of electromagnetic waves of long wavelength in a saturated ferromagnet, taking into account nonlinearity and dispersion. Using a multiscale expansion method, the Maxwell equations in the ferromagnet were reduced to the modified Korteweg–de Vries equation.

In a previous paper [4] we studied the effects of dissipation and nonlinearity on the propagation of a small electromagnetic perturbation in an infinite saturated ferrite, in the presence of an external constant magnetic field, and we showed that such dynamics obeys the Burgers equation in $(1+1)$ and $(2+1)$ dimensions.

In this paper, instead of looking for propagation of waves with long wavelength, we investigate a modulational phenomenon. We will study the way an electromagnetic plane wave is modulated by nonlinear and dispersive effects in an infinite saturated ferromagnet. The main result we will obtain is that the modulation of such waves, in the lowest order of perturbation, is governed by the nonlinear Schrödinger equation (NLS).

In one spatial dimension, it is well known that a dispersive and nonlinear system such as the one studied here can be reduced to the NLS equation. This general result is obtained by using a Taylor expansion of the dispersion relation $\omega(k)$ of the system, which is in this case amplitude dependent, around a wave number k_0 and frequency ω_0. The reasoning involves a hypothesis about the form of the expansion of $\omega(k)$ in power series of the amplitude, which is, from a mathematical point of view, very strong, and makes the whole thing rather heuristic. Furthermore, this derivation of NLS does not give us either the values or the signs of the coefficients in the final NLS equation. The former are necessary to compute explicit solutions, while the latter are necessary in order to characterize the modulational instability of the Benjamin-Feir (or Lighthill) type [5,6]. This instability originates from the explosive growth of side band frequencies and wave numbers adjacent to ω_0 and k_0 in the original wave train.

The difficulties outlined above may be overcome by making use of a perturbation theory to determine explicitly the coefficients in the final NLS equation. We restrict our study to the case of slow modulation (the change of the wave envelope is slow in both space and time in comparison to the carrier wave), which allows us to use the stretched coordinates method, and we consider an infinite system without dissipation in $(1+1)$ dimensions.

To be specific, we will characterize rigorously the instability of the Benjamin-Feir type for electromagnetic waves in a saturated ferro- or ferrimagnetic medium. This is not only important from a theoretical point of view but is also of special interest in experiments and applications. This characterization is carried out by constructing a three-dimensional space of physical parameters, and by determining the regions where the NLS equation admits or does not admit solitonic solutions. In the solitonic case, the wave train is destroyed, bunching into solitons. This is an electromagnetic analog of the well-known Benjamin-Feir instability for the Stokes wave

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train in water waves. This is what we call the focusing case. In the nonsolitonic, defocusing case, only dark-soliton-type solutions of NLS exist, and the wave train is stable and only slowly modulated in amplitude. With a detailed analysis of the dispersion relation, we clarify the relation between its usual representation in the (k,ω) plane and the three new parameters, and interpret the regions obtained in terms of the classical former representation. In this way we determine, *a priori,* the stability or instability of the carrier wave. Furthermore, we found the existence of transition points between the focusing and defocusing regimes (of the carrier wave) in the branch of the dispersion relation corresponding to left elliptically polarized waves. No such point of transition, or stability-instability windows exist for right elliptically polarized waves, which we show to be always stable. It should be remarked that, even in the linear theory, the polarization of a monochromatic wave in the considered medium is entirely determined by the choice of k and one of the three corresponding $\omega(k)$ branches, and that we consider here only monochromatic waves.

In the particular case of longitudinal propagation (circular polarization), another new result is obtained: a plane wave of positive helicity (a left circularly polarized wave in the optical convention) propagating parallel to the applied field is destroyed, converted into solitons, and a plane wave of negative helicity (a right circularly polarized wave) propagates without bunching, being only slowly modulated in its amplitude. In this case there are no points of transition focusing and defocusing and furthermore we can give explicit expressions for the solutions.

This paper is organized as follows: In Sec. II we give the mathematical formulation of the phenomenological model. In Section III, the results of the perturbation scheme are given. In Sec. IV we construct the space of the physical parameters and describe the regions of stability or instability. Also given in this section are reduced explicit expressions for the coefficients of the derived NLS equation, for some particular values of the parameters. In Sec. V we establish, in the dispersion relation, the transition points between the focusing and defocusing regimes. A word on the experimental perspectives of this work is added in Sec. VI, and Sec. VII, the Conclusion, contains a brief note on the first results obtained in $(2+1)$ dimensions on the present matter. Finally, Appendix A contains some mathematical details of the perturbation theory.

II. MATHEMATICAL FORMULATION OF THE PHENOMENOLOGICAL MODEL

In this section we introduce the model our study is based upon. This model, often used in the theoretical or experimental approach to waves in ferrites [1,2], is also sufficient for our purposes. It has the advantage of maximum tractability and it provides a simple phenomenological description of periodic electromagnetic phenomena in a saturated infinite ferromagnet. The range of validity of such a phenomenological description is just that of all theories which involve macroscopic field quantities such as $\mathbf{D},\mathbf{H}$ and the mean field approximation $\mathbf{E},\mathbf{B}$ in the Maxwell equation. It is based on the general form of Maxwell's equations in the infinite ferrite, which in MKS units reads

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\ , \tag{1}$$

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}\ . \tag{2}$$

In (1) and (2), $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ have their standard meanings. The constitutive equations in the ferromagnetic for $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{H},\mathbf{B}$ are given by

$$\mathbf{D}=\hat{\epsilon}\mathbf{E}\ , \tag{3}$$

$$\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})\ , \tag{4}$$

where we shall assume that $\hat{\epsilon}$ is the scalar permittivity of a ferromagnet, μ_0 is the magnetic permeability in the vacuum, and $\mathbf{M}$ is the magnetization density in a ferromagnet. We consider a ferromagnet with saturated magnetization density. In [7] it was shown that the dynamic equation for $\mathbf{M}$ when damping is neglected is

$$\frac{\partial\mathbf{M}}{\partial t}=-\mu_0\delta\mathbf{M}\times\mathbf{H}_{\text{eff}}\ , \tag{5a}$$

where δ is the gyromagnetic ratio and $\mathbf{H}_{\text{eff}}$ is

$$\mathbf{H}_{\text{eff}}=\mathbf{H}+\beta\mathbf{n}(\mathbf{n}\cdot\mathbf{M})+\alpha\nabla^2\mathbf{M}\ .$$

The terms $-\mu_0\delta\beta\mathbf{M}\times\mathbf{n}(\mathbf{n}\cdot\mathbf{M})$ and $-\mu_0\delta\alpha\mathbf{M}\times\nabla^2\mathbf{M}$ represent the effect of magnetic anisotropy and inhomogeneous exchange interaction, respectively. We may here neglect these terms: the first one simply by assuming our medium to be isotropic and the second one because we are considering electromagnetic waves and the space scales associated with them substantially exceed the characteristic length of the inhomogeneous exchange interaction (given by α). This last one is very important when we consider spin waves, but this is not the case here. For this reason we will work with the equation

$$\frac{\partial\mathbf{M}}{\partial t}=-\mu_0\delta\mathbf{M}\times\mathbf{H}\ . \tag{5b}$$

The approximation of (5a) by (5b) has been used by most authors who studied such problems [1,2].

Equation (5b) shows that $\mathbf{M}$ is not parallel to $\mathbf{H}$ and that it is nonlinearly related to $\mathbf{H}$.

Taking the curl of Eq. (2) and using (1), (3), and (4), we have

$$-\nabla(\nabla\cdot\mathbf{H}+\nabla^2\mathbf{H}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{H}+\mathbf{M})\ , \tag{6}$$

where $c=1/\sqrt{\hat{\epsilon}\mu_0}$ is the speed of light based on the dielectric constant of the ferromagnet. If the magnetization were zero, $\nabla\cdot\mathbf{H}=0$ and (6) would be the linear wave equation, satisfied by isotropic, dispersionless transverse waves propagating at speed c. Such is not the case and Eqs. (5b) and (6) are a system of nonlinear partial differential equations for $\mathbf{M}$ and $\mathbf{H}$. All linear theories are based on Eq. (6) and a linearized version of (5b) which is obtained considering that **H** is composed of an applied constant field plus a superposed small alternating (rf) field of constant amplitude. We do not make such a hypothesis and we are going to consider the harmonic solutions of (5b) and (6) in one space coordinate x and time t which, although of small amplitude, are nevertheless large enough so that the effect of nonlinearity cannot be neglected. Nonlinear terms give rise to a modulation of the amplitude as well as waves of higher harmonics. Our aim is to investigate how the amplitude is modulated by nonlinear effects with the condition that this modulation is slow compared to the period of the oscillations of the sinusoidal part.

III. PERTURBATION SCHEME, NONLINEAR SCHRÖDINGER EQUATION, AND THE ASSOCIATED BENJAMIN-FEIR INSTABILITY

Let us seek a solution of (5b) and (6) under the form of a Fourier expansion in harmonics of the fundamental $E=\exp\{i(kx-\omega t)\}$ as

$$\mathbf{M}=\sum_{n=-\infty}^{+\infty}\mathbf{M}^nE^n , \tag{7a}$$

$$\mathbf{H}=\sum_{n=-\infty}^{+\infty}\mathbf{H}^nE^n , \tag{7b}$$

where the Fourier components are developed in a Taylor series in powers of the small parameter ϵ measuring the normalized amplitude of the applied rf field:

$$\mathbf{M}^n=\sum_{j=0}^{\infty}\epsilon^j\mathbf{M}_j^n(\xi,\tau) , \tag{8a}$$

$$\mathbf{H}^n=\sum_{j=0}^{\infty}\epsilon^j\mathbf{H}_j^n(\xi,\tau) . \tag{8b}$$

Here above we have the reality conditions $\mathbf{M}^{-n}=(\mathbf{M}^n)^*$ and $\mathbf{H}^{-n}=(\mathbf{H}^n)^*$, where the asterisk denotes complex conjugation and τ,ξ are slow variables introduced through the stretching

$$\tau=\epsilon^2t , \tag{9a}$$

$$\xi=\epsilon(x-Vt) , \tag{9b}$$

where the velocity V will be determined later as a solvability condition of Eqs. (5b) and (6). The expansions (7) and (8) include fast local oscillations through the dependence on the harmonics E^n and slow variation (modulation) in amplitude taken into account by the τ,ξ dependence of $\mathbf{M}_j^n$ and $\mathbf{H}_j^n$. The amplitude of the wave is thus assumed to vary along the direction of propagation only. The effect of transverse variation is not studied here. A brief note on the preliminary results we have already obtained on this subject may be found at the end of this paper (Sec. VII). Substituting (7a) and (7b) in (5b) and (6) for $\mathbf{M}=(M_x,M_y,M_z)$ and $\mathbf{H}=(H_x,H_y,H_z)$; rescaling **M**, **H** and t into $\delta\mu_0\mathbf{M}/c$, $\delta\mu_0\mathbf{H}/c$, and ct; and collecting powers of E, we obtain

$$\left[\frac{\partial}{\partial t}-in\omega\right]\mathbf{M}^n=-\sum_{p+q=n}\mathbf{M}^p\times\mathbf{H}^q , \tag{10a}$$

$$\left[\frac{\partial^2}{\partial t^2}-2in\omega\frac{\partial}{\partial t}-n^2\omega^2\right](H_s^n+M_s^n)=\left[\frac{\partial^2}{\partial x^2}+2ink\frac{\partial}{\partial x}-n^2k^2\right]H_s^n(1-\delta_{s,x}) , \tag{10b}$$

where $s=x$, y, and z and $\delta_{s,i}$ is the Kronecker symbol ($\delta_{s,i}=1$ for $s=i$ and 0 otherwise). Introducing now the expansions (8) and the slow variables (9) into (10), we may proceed to collect and solve different orders ϵ^j and harmonics n [order (j,n)] with the following assumptions: we suppose that $\mathbf{M}_0^0=\mathbf{m}$ and $\mathbf{H}_0^0$ are constants and that $\mathbf{M}_0^n,\mathbf{H}_0^n=\mathbf{0}$ for $n\neq0$. The assumed conditions at infinity are $M_j^n,H_j^n\underset{\xi\to-\infty}{\longrightarrow}0$ for $j\neq0$, and all n except for $(j,|n|)=(1,1)$, where the limit is assumed to be a finite constant. The field $\mathbf{H}_0^0$ represents the external constant magnetic field where the ferrite is immersed and $\mathbf{M}_0^0=\mathbf{m}$ is close to the magnetization of saturation. The state $\mathbf{H}_0^0,\mathbf{M}_0^0$ represents an initial static state and the following terms of their developments (8a) and (8b), a perturbation of this state. For an appropriate choice of the Cartesian coordinate system, we may write $\mathbf{m}=(m_x,m_t,0)$. We obtain the following results (the details of their derivation are in Appendix A) for different orders (j,n). For $j=0$ we find that $\mathbf{H}_0^0$ is necessarily colinear to **m** and define α such that $\mathbf{H}_0^0=\alpha\mathbf{m}$. For $j=1$ and $n>1$ we can show that

$$\mathbf{H}_1^n=\mathbf{M}_1^n=\mathbf{0} . \tag{11}$$

For $(j,n)=1,1)$ we have

$$M_{1,x}=-H_{1,x}^1=-i\gamma\mu m_tg(\xi,\tau) , \tag{12a}$$

$$M_{1,y}^1=-\gamma H_{1,y}^1=i\gamma\mu m_xg(\xi,\tau) , \tag{12b}$$

$$M_{1,z}^1=-\gamma H_{1,z}^1=-\gamma^2\omega g(\xi,\tau) , \tag{12c}$$

where γ and μ are two dependent parameters given by

$$\gamma=1-\frac{k^2}{\omega^2} , \tag{13}$$

$$\mu=1+\alpha\gamma . \tag{14}$$

The function $g(\xi,\tau)$ is arbitrary and such that $\lim_{\xi\to\infty}|g(\xi,\tau)|^2$ is a finite constant that we will call λ, ($\lambda\neq0$). The expressions (12a), (12b), and (12c) are obtained under the condition that ω verifies the dispersion relation

$$(\omega^2-k^2)[(1+\alpha)\omega^2-\alpha k^2](1+\alpha)m_t^2+[(1+\alpha)\omega^2-\alpha k^2]^2m_x^2=(\omega^2-k^2)^2\omega^2 . \tag{15}$$

The very important and useful relation giving k as a function of ω (which we will use in Secs. IV and V) is obtained from (15) and reads

$$k_{\pm}=\omega\left[\frac{(1+\alpha)(2\alpha+\sin^2\varphi)-2\nu^2\mp\sqrt{(1+\alpha)^2\sin^4\varphi+4\nu^2\cos^2\varphi}}{2[\alpha(\alpha+\sin^2\varphi)-\nu^2]}\right]^{1/2}, \tag{16}$$

where $\nu=\omega/m$ and φ are defined by $m_x=m\cos\varphi$ and $m_t=m\sin\varphi$. These two possible values of k represent two elliptically polarized waves propagating in the same direction but with different velocities.

We now make the hypothesis that the pulsation of the applied monochromatic rf wave is one of the three solutions $\omega(k)$ of (15). Thus, the group velocity of the primary progressive wave reads

$$V_g=\frac{\partial\omega(k)}{\partial k}=\frac{(b+1)}{\gamma\mu u^2+b+1}u, \tag{17}$$

where $u=\omega/k$ is the phase velocity and the parameter b is given by

$$b=\frac{\mu^2 m_x^2}{\gamma^2\omega^2}. \tag{18}$$

For $(j,n)=(1,0)$, we have

$$\mathbf{H}_1^0=\mathbf{M}_1^0=\mathbf{0}, \tag{19}$$

and this completes the results at order $(1,n)$ for all n. Using these results, we can determine the orders $(2,n)$ for all n. For $|n|>2$, we show that

$$\mathbf{H}_2^n=\mathbf{M}_2^n=\mathbf{0}. \tag{20}$$

The order (2,2) is given by

$$M_{2,x}^2=-H_{2,x}^2=-\frac{\gamma\mu^2 m_x(1-b)}{2u^2(1+\alpha)}g^2, \tag{21a}$$

$$M_{2,y}^2=-\gamma H_{2,y}^2=\frac{\gamma^2\mu m_t}{2u^2}(1+b)g^2, \tag{21b}$$

$$M_{2,z}^2=-\gamma H_{2,z}^2=\frac{i\gamma\mu^2 m_x m_t}{u^2\omega}g^2. \tag{21c}$$

For (2,1), we obtain expressions for the $\mathbf{H}_2^1,\mathbf{M}_2^1$ components. They read

$$H_{2,x}^1=-M_{2,x}^1=-\gamma\mu m_t f+\Omega m_t(b+1+2\alpha\gamma)\frac{\partial g}{\partial\xi}, \tag{22a}$$

$$H_{2,y}^1=\mu m_x f+\Omega\frac{m_x}{\gamma}(1-b)\frac{\partial g}{\partial\xi}, \tag{22b}$$

$$H_{2,z}^1=i\gamma\omega f, \tag{22c}$$

$$M_{2,y}^1=-\gamma\mu m_x f+\Omega m_x(b+1+2\alpha\gamma)\frac{\partial g}{\partial\xi}, \tag{22d}$$

$$M_{2,z}^1=-i\gamma^2\omega f+2i\gamma\omega\Omega\frac{\partial g}{\partial\xi}, \tag{22e}$$

where $f(\xi,\tau)$ is an arbitrary function and Ω is

$$\Omega=\frac{k^2}{\omega^3}(V-u). \tag{23}$$

The expressions (22a)–(22e) are obtained under some solvability condition which determines V in Eq. (9b). It reads

$$V=V_g=\frac{\partial\omega}{\partial k}. \tag{24}$$

For the terms of order (2,0), we obtain

$$H_{2,x}^0=-M_{2,x}^0=m_x(1+\alpha\beta)\Phi, \tag{25a}$$

$$H_{2,y}^0=-\frac{1}{\beta}M_{2,y}^0=m_t(1+\alpha)\Phi-\frac{2\gamma(1-\gamma)}{1+\alpha\beta}\mu^2 m_t|g|^2, \tag{25b}$$

$$H_{2,z}^0=M_{2,z}^0=0, \tag{25c}$$

where Φ is a function of (ξ,τ) which will be determined below and β is the parameter

$$\beta=1-\frac{1}{V^2}. \tag{26}$$

The next order $(3,n)$ is the most laborious, and it is the one which allows us to find the function $\Phi(\xi,\tau)$ and the nonlinear evolution of $g(\xi,\tau)$. At the order (3,0), we obtain the equation determining $\Phi(\xi,\tau)$. It reads

$$\Phi(\xi,\tau)=\frac{1}{d}\left[\frac{2\beta\gamma(1-\gamma)\mu^2}{1+\alpha\beta}m_t^2-\frac{\gamma\omega\Omega}{V}\Lambda\mu\right]|g|^2+\frac{\gamma\omega\Omega\Lambda\mu\lambda}{Vd}, \tag{27}$$

with

$$d=m_x^2(1+\alpha\beta)+(1+\alpha)\beta m_t^2, \tag{28}$$

$$\Lambda=\frac{\Gamma}{\gamma\mu^2(1+\alpha)}[2b\gamma(1+\alpha)+2\mu-(b^2+1)(1-\gamma)], \tag{29}$$

$$\Gamma=\gamma^2\omega^2. \tag{30}$$

At the order (3,1), we obtain a compatibility condition which gives a nonlinear evolution equation for $g(\xi,\tau)$ [the term f coming from $\mathbf{H}_2^1,\mathbf{M}_2^1$ disappears using (15)]. It reads

$$iA\frac{\partial g}{\partial\tau}+B\frac{\partial^2 g}{\partial\xi^2}+Cg|g|^2+D\lambda g=0, \tag{31}$$

with the condition $|g|^2\to\lambda$ for $\xi\to-\infty$, and where the real constants A, B, C, and D are given by

$$A=-\frac{2\Gamma\omega}{\mu u^2}(b+1+\gamma\mu u^2), \tag{32}$$

$$B=\frac{\Gamma\gamma u^2}{(b+1+\gamma\mu u^2)^2}\mathcal{P}\ , \tag{33}$$

$$C=\frac{\Gamma^2\mu}{2(1+\alpha)}\frac{\mathcal{L}}{\mathcal{Q}}\ , \tag{34}$$

$$D=+\frac{\Gamma^2\mu}{1+\alpha}[2\mu-(b+1)(1-\gamma)]\frac{\mathcal{H}}{\mathcal{Q}}\ , \tag{35}$$

with $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{L}$ given by

$$\mathcal{P}=3b^2-b-(b+1)(3\alpha\gamma+\mu u^2)\ , \tag{36}$$

$$\mathcal{Q}=(b+1)^2-[2(b+1)+\gamma\mu u^2](\mu-b)\ , \tag{37}$$

$$\mathcal{H}=(1-\gamma)(1-3b)(b+1)^2+\gamma[2(b+1)+\gamma\mu u^2]\times[(\alpha+1)(b+1)+\mu(1-3b)]\ , \tag{38}$$

$$\mathcal{L}=(b+1)^2\mathcal{B}_1-[2(b+1)+\gamma\mu u^2]\mathcal{B}_2\ , \tag{39}$$

with

$$\mathcal{B}_1=(1-\gamma)\{(1-\gamma)[15b^2-6b-1]+4\mu(1-3b)\}\ , \tag{40}$$

$$\mathcal{B}_2=2\gamma(b+1)\{(1-\gamma)[(1-3b)\mu+(1+\alpha)(b+1)]-2\mu(1+\alpha)\}+(1-\gamma)(1-b)\{-3b^2(1-\gamma)+b[(1-\gamma)(3\mu-5)+4(3\gamma-1)\mu]+\mu(1-\gamma)\}-4\gamma\mu^2(1-3b)\ . \tag{41}$$

We can now make the transformation

$$g(\xi,\tau)=\varphi(\xi,\tau)e^{i(D/A)\lambda\tau}\ , \tag{42}$$

$$T=\frac{B}{A}\tau\ ,\quad X=\xi\ ,\quad E=\frac{C}{B}\ , \tag{43}$$

and we obtain

$$i\varphi_T+\varphi_{XX}+E\varphi|\varphi|^2=0\ . \tag{44}$$

Equation (31) [or Eq. (44)] is the nonlinear Schrödinger equation [8], which appears in many branches of physics when nonlinear modulation of waves is studied. This equation has been extensively studied by several methods and we know that it belongs to the class of soliton equations.

The nature of solutions of the NLS as well as its physical meaning depend drastically on the sign of the product BC (or E) [9]. For $BC>0$ we know that the incident carrier wave is destroyed by nonlinearity and dispersion and it bunches into solitons (the focusing case). For $BC<0$ the incident carrier wave evolves without bunching in a self-similar form. These two cases characterize regions in the space of the physical parameters (scalar permittivity, magnetic permeability, gyromagnetic ratio, values of the dc applied field, frequency of the rf field etc.) where stability or instability of the incident carrier wave occurs. This instability of electromagnetic propagation in a saturated ferrite is reminiscent of the Benjamin-Feir-instability phenomenon of the Stokes wave train, in water-wave theory [5,6].

IV. THE SPACE OF PARAMETERS u, α, AND b: REGIONS OF STABILITY OR INSTABILITY FOR THE WAVE TRAIN

The BC product in (31) can be written as

$$BC=\theta\gamma\mu\frac{\mathcal{L}\mathcal{P}}{\mathcal{Q}}\ , \tag{45}$$

with θ always positive, given by

$$\theta=\frac{\Gamma^3u^2}{2(1+\alpha)(b+1+\gamma\mu u^2)^2}\ . \tag{46}$$

The sign of BC is determined by the product $\gamma\mu\mathcal{L}P/\mathcal{Q}$ through the values of the three positive parameters

$$u=\frac{\omega}{k}\ , \tag{47}$$

$$\alpha=\frac{|\mathbf{H}_0^0|}{|\mathbf{M}_0^0|}\ , \tag{48}$$

$$b=\frac{\mu^2m^2\cos^2\varphi}{\gamma^2\omega^2}\ . \tag{49}$$

The first one is the phase velocity, the second one measures the relative intensity between the external (constant) magnetic field ($\mathbf{H}_0^0$) and the magnetization of saturation ($\mathbf{M}_0^0$), and the last one is related to the angle φ between the direction of propagation of the carrier wave and the external magnetic field $\mathbf{H}_0^0$. The complexity of expressions (32) to (41) prevents us from expressing the results in terms of φ (or $\cos\varphi$) itself as a third variable, and we must introduce the "auxiliary" variable b given by Eq. (49) to be able to achieve the discussion. Note that b depends also on α and u.

There are several regions in the (b,u,α) space where BC is positive, zero, or negative. In general we are constrained to make a numerical and approximate study of the expressions of B and C to determine this sign. For a few very particular values or limits of one of the three parameters u, α, and b only, we are able to calculate exactly the sign of BC as a function of the two parameters left. We will show in this section the results concerning the sign of BC in a plane (b,u) with α given.

Expressions (33) and (34) give us B and C exactly. However, because of the high complexity of these expressions, we cannot obtain the sign of BC as an explicit function of u, α, and b and, consequently, we cannot establish exactly the regions of transition between the stable and unstable regimes. In order to get these regions, we have to make an analytical and numerical study of the expressions of B and C. We divide the presentation of our results into three groups according to their nature: the first one, contained in Sec. IV A, corresponds to the results obtained through an analytical and, mainly, numerical analysis of the expressions of B and C. Figure 2, at the end of this subsection, summarizes the results on the BC sign as a function of u, α, and b. The second group, given in Sec IV B, corresponds to the cases where we obtain approximative explicit analytical expressions for B and C and, subsequently, the sign of BC. Finally, Sec. IV C, contains the very important case for which we can calculate exactly and explicitly the values of A, B, C and the sign of the BC product as a function of u, α and φ itself.

A. The sign of the BC product: an analytical and numerical study

We must first determine range of variation of (b,u,α) when the physical parameters take all their possible values. The magnetic field $\mathbf{H}_0^0$ and magnetization density $\mathbf{M}_0^0$ have the same direction, so that $\alpha>0$ (the other solution is unstable). α takes thus all positive real values. We can choose u positive, and an elementary study of the three branches $\omega(k)$ of the dispersion relation (15) shows that u takes all positive real values except in the interval

$$\left[\sqrt{\alpha/(1+\alpha)}\ ,\ \sqrt{(\alpha+\sin^2\varphi)/(1+\alpha)}\right] .$$

It remains to determine the allowed values of b (as a function of α and u). $b=\mu^2 m_x^2/\gamma^2\omega^2$ is a positive number or zero.

From the dispersion relation (15), which can be written as

$$\gamma\mu(1+\alpha)m_t^2=(1-b)\Gamma ,$$

$$\Gamma=\gamma^2\omega^2 ,$$

we see that the sign and the values of b are determined from those of u, through the values of γ $(\gamma=1-1/u^2)$ and μ $[\mu=1+\alpha(1-1/u^2)]$ with $\alpha>0$ given. We see that if

$$\gamma\mu>0\ ,\ \text{then } b\in]0,1[\ ,$$

and if

$$\gamma\mu<0\ ,\ \text{then } b\in]1,\infty[\ .$$

Table I summarizes the results of the sign of $\gamma\mu$ and the values of b as a function of the values of u.

Now we consider the expression of $\mathcal{Q}$, $\mathcal{P}$, and $\mathcal{L}$ [given by (37), (36), and (39)] as being functions of (b,u) (for α given) with (b,u) two free parameters which can take all the values between zero and infinite, independently of one another. This is a matter of convenience and the final results on BC's sign given in Fig. 2 will be carried out considering the preceding results.

Sign of $\mathcal{Q}(b,u,\alpha)$. For α fixed, the shape of the curve $\mathcal{Q}(b,u)=0$ is obtained numerically with the aid of the following analytical results (we do not write α explicitly in the arguments of $\mathcal{Q}$, $\mathcal{P}$, . . . , when it is not necessary).

For $b=0$,

$$\mathcal{Q}(0,u)=-u^2(1+2\alpha\gamma+\alpha^2\gamma^3)$$

becomes zero only for $u=u_1\in]\sqrt{\alpha/(1+\alpha)},1[$.

For $b=1$,

$$\mathcal{Q}(1,u)=u^2[-\alpha^2\gamma^3-\alpha\gamma(4-3\gamma)+4-4\gamma]$$

becomes zero only for $u=v_1$, such that

$$\alpha=\frac{3\gamma_1-4+\sqrt{16-8\gamma_1-7\gamma_1^2}}{2\gamma_1^2} ,$$

where $\gamma_1=1-1/v_1^2$.

For $u=1$, $\mathcal{Q}=(b+1)(3b-1)$ becomes zero only if $b=\frac{1}{3}$.

For $u\to\infty$,

$$\mathcal{Q}\sim-(1+\alpha)u^2(1+\alpha-b)$$

becomes zero only for $b=1+\alpha$.

These results are summarized in Fig. 1 (dashed line). At the left (right) of the curve $\mathcal{Q}(b,u)=0$, this function is positive (negative).

Sign of $\mathcal{P}(b,u,\alpha)$. Identically, as for $\mathcal{Q}$, the shape of the curve $\mathcal{P}(b,u)=0$ is obtained numerically, considering that

For $b=0$,

$$\mathcal{P}(0,u)=-u^2(1+4\alpha\gamma-3\alpha\gamma^2)\mathcal{P}(b,u)=0$$

has a unique solution

$$u=u_2\in]\sqrt{\alpha/(1+\alpha)}\ ,1[\ .$$

For $u=1$,

$$\mathcal{P}(b,1)=(b-1)(3b+1)$$

becomes zero only if $b=1$. Figure 1 contains these re-

TABLE I. The sign of the product $\gamma\mu$ for $u\in]0,\infty[$ and the corresponding values of b.

Values of u	$u\in]0,\left[\frac{\alpha}{1+\alpha}\right]^{1/2}[$	$u=\left[\frac{\alpha}{1+\alpha}\right]^{1/2}$	$u\in]\left[\frac{\alpha}{1+\alpha}\right]^{1/2},1[$	$u=1$	$u=]1,\infty[$
Sign of γ	$-$	$-$	$-$	$\gamma=0$	$+$
Sign of μ	$-$	$\mu=0$	$+$	$+$	$+$
Sign of $\gamma\mu$	$+$	$\gamma\mu=0$	$-$	$\gamma\mu=0$	$+$
Values of b	$b\in]0,1[$	$b=0$	$b\in]1,\infty[$	b not defined	$b\in]0,1[$

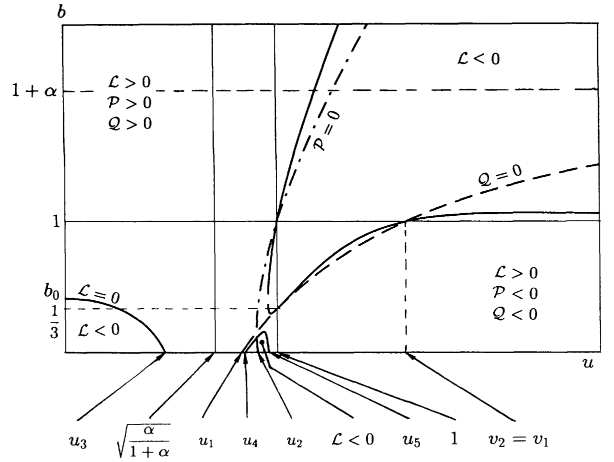

FIG. 1. Curves $\mathcal{Q}(b,u)=0$ (dashed line), $\mathcal{P}(b,u)=0$ (dot-dashed line) and $\mathcal{L}(b,u)=0$ (solid line), with an indication of the sign of $\mathcal{Q}(b,u)$, $\mathcal{P}(b,u)$, and $\mathcal{L}(b,u)$ ($\alpha=1$).

sults also (dot-dashed line).

Sign of $\mathcal{L}(b,u,\alpha)$. The curve defined in the quarter of plane $b>0$, $u>0$ by $\mathcal{L}(b,u,\alpha)=0$ (for a fixed α) presents three branches. We begin by giving the zeros of $\mathcal{L}$ for same particular values of u.

For $u=1$, $\mathcal{L}=(b^2-1)(9b^2-1)$ becomes 0 for $b=1,\frac{1}{3}$.

For $u\to 0$,

$$\mathcal{L}\sim-\alpha^2\gamma^4[15b^2-6(1-2\alpha)b-1-4\alpha]$$

becomes zero for

$$b=b_0(\alpha)=\frac{3(1-2\alpha)+2\sqrt{3(2+2\alpha+3\alpha^2)}}{15} . \tag{50}$$

For $u\to\infty$,

$$\mathcal{L}(b,u)\sim 8(1+\alpha)(1-b)u^2$$

becomes zero if $b=1$. Now we analyze the zeros of $\mathcal{L}$ for some particular values of b. In the case $b=0$, we obtain

$$\mathcal{L}(0,u)=(1-\gamma)[3+\gamma(4\alpha+1)]$$

$$+\frac{1}{\gamma-1}(\gamma-2-\alpha\gamma^2)[-1+6\gamma+\alpha\gamma+3\gamma^2$$

$$+14\alpha\gamma^2+4\alpha^2\gamma^2+\alpha\gamma^3+4\alpha^2\gamma^3] .$$

The expression $\mathcal{L}(0,u)$ for $u\to 0$, $u=\sqrt{\alpha/(1+\alpha)}$, and $u=1$ becomes

$$\mathcal{L}(0,u)\underset{u\to 0}{\sim}-\alpha^2\gamma^4(1+4\alpha)<0 ,$$

$$\mathcal{L}(0,\sqrt{\alpha/1+\alpha})=\frac{3(1+\alpha)^2}{\alpha^2}>0 ,$$

$$\mathcal{L}(0,1)=1>0 .$$

We see that $\mathcal{L}(0,u)=0$ has an odd number of roots for $u\in]0,\sqrt{\alpha/(1+\alpha)}[$ and an even number of roots for $u\in]\sqrt{\alpha/(1+\alpha)},1[$. Numerically, we observe that there is one root u_3 for $u\in]0,\sqrt{\alpha/(1+\alpha)}[$, and two roots u_4,u_5 ($u_4<u_5$) for $u\in]\sqrt{\alpha/(1+\alpha)},1[$.

We can only know the asymptotic values of u_3, u_4, and u_5 for $\alpha\gg 0$. They read

$$u_3\simeq\frac{1}{\sqrt{2}}\left[1-\frac{3}{4\alpha}\right] ,$$

$$u_4\simeq 1-\frac{1}{4\alpha} ,$$

$$u_5\simeq 1-\frac{1}{8\alpha} .$$

We also give here the expression of the curve joining $b_0(\alpha)$ to u_3 for $\alpha\gg 1$, which will be used later. It reads

$$b=1-\frac{2}{3-2u^2} .$$

Moreover, we can prove that the branches do not cross the line $u=\sqrt{\alpha/(1+\alpha)}$ for any value of b.

For $b=1$,

$$\mathcal{L}(1,u)=8(1+\alpha)\gamma[\alpha^2\gamma^3+\alpha\gamma(4-3\gamma)+4\gamma-4]$$

becomes zero for the same value of u which satisfies $\mathcal{Q}(1,u)=0$, i.e., $u=v_1$. Figure 1 contains all the numerical and analytical results about $\mathcal{L}(b,u)=0$ (solid line).

Now we are in a position to calculate numerically the sign of BC. Bearing in mind that $BC=\theta\gamma\mu(\mathcal{P}\mathcal{L}/\mathcal{Q})$, with $\theta>0$, and using the results of Fig. 1 and Table I, we arrive at the final results on the BC sign, as a function of (b,u) with α fixed. These results are summarized in Fig. 2. The dotted areas are prohibited regions in the (b,u,α) space. (The fact that $u\notin[\sqrt{\alpha/(1+\alpha)},\sqrt{(\alpha+\sin^2\varphi)/(1+\alpha)}]$ is not represented.) The signs of BC are indicated inside each allowed region in this plane. In the regions where $BC>0$, in-going plane waves are unstable and in those where $BC<0$, waves are stable. This figure summarizes the first main result of our work: having the initial physical parameters $|\mathbf{H}_0^0|$, $|\mathbf{M}_0^0|$, ω, k, and φ, we can place the corresponding point in the (b,u) plane [using (47)–(49)] and know if we are in a region of stability or unstability for the carrier wave.

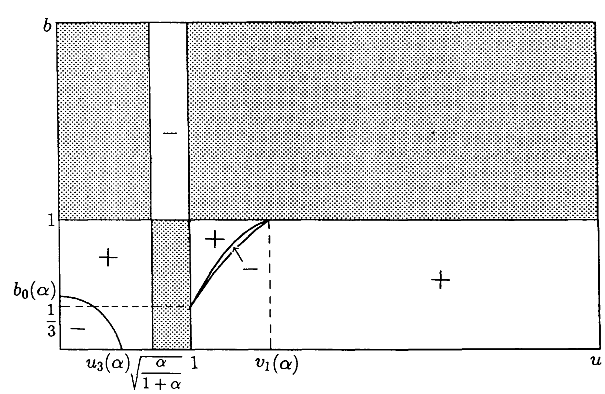

FIG. 2. The sign of BC in the (b,u) plane for α given ($\alpha=1$).

B. The sign of the BC product: analytical and approximate expressions for A, B, and C

The complete solution concerning the sign of BC is given in Fig. 2. But it is mainly a numerical solution, and the problem of the enormous complexity of the coefficients A, B, and C remains. In particular, formal and easier explicit expressions for A, B, and C and the associated BC sign, as functions of the physical parameters, even if they are approximate, are interesting, for example, for constructing explicitly the soliton or the dark-soliton solution. We have calculated thus A, B, and C in three particular cases.

The case $\alpha\to\infty$, ω fixed. This is an important case of a physical point of view because $\alpha\to\infty$ means that the external magnetic field $|\mathbf{H}_0^0|$ is strong ($\alpha=|\mathbf{H}_0^0|/|\mathbf{M}_0^0|$) in relation to the magnetization of saturation $|\mathbf{M}_0^0|$. There are two subcases that are easy to study by making $\gamma=a/\alpha$ in the dispersion relation

$$(1+a)^2 m_x^2+\frac{a}{\alpha}(1+a)(1+\alpha)m_t^2=\frac{a^2}{\alpha^2}\omega^2 .$$

For $\alpha\to\infty$ with ω fixed, we obtain two solutions for a:

$$a=-\cos^2\varphi \quad\text{and}\quad a=-1 .$$

For $a=-\cos^2\varphi$, we have

$$\gamma\sim\frac{-\cos^2\varphi}{\alpha}\ ,\quad u=1-\frac{\cos^2\varphi}{2\alpha}+O(1/\alpha)\ ,$$

$$b\sim\frac{m^2\alpha^2\sin^4\varphi}{\omega^2\cos^2\varphi}$$

and φ such that $m_x=|\mathbf{m}|\cos\varphi$, $\mathbf{m}=\mathbf{M}_0^0$, and A,B, and C given by

$$A\sim-2m^2\omega\cos^2\varphi\sin^2\varphi\ ,$$

$$B\sim-3\frac{\omega^2}{\alpha^2}\cos^6\varphi\ ,$$

$$C\sim\frac{3}{2}\frac{m^4}{\alpha}\cos^4\varphi\sin^{10}\varphi\ .$$

Thus $BC<0$. Since $u\to1$, $b\to+\infty$, this subcase is situated near to the line $u=1$, and at his left, into the central upper domain marked by a minus sign in Fig. 2.

For $a=-1$ we have

$$\gamma=-\frac{1}{\alpha}-\frac{\omega^2}{n^2\alpha^3\sin^2\varphi}+O\left[\frac{1}{\alpha^3}\right]\ ,$$

$$u\sim1-\frac{1}{2\alpha}\ ,$$

$$b\sim\frac{\omega^2\cos^2\varphi}{m^2\alpha^2\sin^4\varphi}\ ,$$

with A, B, and C given by

$$A\sim2m^2\omega\sin^2\varphi\ ,$$

$$B\sim-3\frac{\omega^2}{\alpha^3}\ ,$$

$$C\sim-\frac{3\omega^6}{2\alpha^7m^2\sin^2\varphi}\ .$$

Thus $BC>0$ and since $b\to0$, $u\to1$ [with $u<\sqrt{\alpha/(1+\alpha)}$ because for $\gamma=-1/\alpha$, $u=\sqrt{\alpha/(1+\alpha)}$], this subcase is situated near and left to the line $u=\sqrt{\alpha/(1+\alpha)}$ into the region $BC>0$ in Fig. 2.

The case $\alpha\to\infty$, b and u fixed. From the dispersion relation and considering that $\mu\sim\alpha\gamma$, for $\alpha\to\infty$, we obtain

$$\omega\sim\alpha m\ ,\quad b\sim\cos^2\varphi\ ,$$

and A, B, and C are given by

$$A\sim-2\alpha^3\gamma^3m^3\ ,$$

$$B\sim-\frac{\alpha m^2}{u^2}(b+1)(3+u^2)\ ,$$

$$C\sim-2m^4\alpha^5\gamma^6[b+1+\gamma(1-3b)]\ .$$

Thus, the sign of BC is governed by the quantity $[b+1+\gamma(1-3b)]$. Thus, if $u\geq1/\sqrt{2}$, $BC>0$ for all $b\in\,]0,+\infty[$, and if $u<1/\sqrt{2}$, $BC>0$ for

$$b\in\,]1-2/(3-2u^2),+\infty[.$$

The case $u\to0$. In this case we have (with m_x, m_t, and α constant)

$$\omega\sim m\sqrt{\alpha(\alpha+\sin^2\varphi)}\ ,\quad b\to\frac{\alpha\cos^2\varphi}{\alpha+\sin^2\varphi}\ ,$$

$$A\sim\frac{2m^3}{u^6}[\alpha(\alpha+\sin^2\varphi)]^{3/2}\ ,$$

$$B\sim\frac{-3m^2}{u^2}[2\alpha+\sin^2\varphi(1-\alpha)]\ ,$$

$$C\sim\frac{\alpha^3m^4}{2u^{14}}(1+\alpha)F\ ,$$

where

$$F=4\alpha^3(1-3\cos^2\varphi)+\alpha^2(9-14\cos^2\varphi-3\cos^4\varphi)$$
$$+2\alpha\sin^2\varphi(3+\cos^2\varphi)+\sin^4\varphi\ .$$

To determine the sign of BC, it is more convenient to use the following expression of C:

$$C\sim\frac{\alpha^3m^4}{2u^{14}}\frac{(\alpha+\sin^2\varphi)^2}{1+\alpha}f\ ,$$

with f given by

$$f=4\alpha(1-3b)-15b^2+6b+1\ .$$

Thus BC has the sign of $-f$:

$$BC>0\quad\text{if } b>b_0(\alpha)$$

and

$$BC<0\quad\text{if } b<b_0(\alpha)\ .$$

The result may be written explicitly in terms of φ:

$$BC>0 \text{ if and only if } \cos\varphi > \left[\frac{(\alpha+1)b_0(\alpha)}{\alpha+b_0(\alpha)}\right]^{1/2},$$

with $b_0(\alpha)$ given by (50).

C. The sign of the BC product: exact analytical solution for longitudinal propagation

Let us consider now the only case having an analytical exact solution. It is represented by the straight line $b=1$ in Fig. 2 and it corresponds to $\varphi=0$: the direction of propagation of the incident wave is parallel to the external magnetic field $\mathbf{H}_0^0$. This case is very useful in practical situations. Putting $m_x=m$, $m_t=0$, $\varphi=0$, and $\epsilon=\pm 1$, a parameter indicating the two possible polarizations of the wave, and using the parameter $\nu=\omega/m$, the dispersion relation (16) may be written as follows:

$$k_\epsilon=\omega\left[\frac{\nu+\epsilon(1+\alpha)}{\nu+\epsilon\alpha}\right]^{1/2}. \tag{51}$$

We also know that

$$u=\left[\frac{\nu+\epsilon\alpha}{\nu+\epsilon(1+\alpha)}\right]^{1/2}, \tag{52}$$

$$\gamma=\frac{-1}{\alpha+\epsilon\nu}, \tag{53}$$

$$\mu=\frac{\epsilon\nu}{\alpha+\epsilon\nu}. \tag{54}$$

Consequently, the coefficients A, B, and C are exactly given by

$$A=-\frac{2\epsilon m^3\nu^2}{(\alpha+\epsilon\nu)^3}[2(\alpha+\epsilon\nu)(\alpha+\epsilon\nu+1)-\epsilon\nu], \tag{55a}$$

$$B=-2m^2\nu^2\frac{[(\alpha+\epsilon\nu)(1+4\alpha)+3\alpha]}{(\alpha+\epsilon\nu)[2(\alpha+\epsilon\nu)(\alpha+\epsilon\nu+1)-\epsilon\nu]^2}, \tag{55b}$$

$$C=4\epsilon m^4\nu^5\frac{(\alpha+\epsilon\nu+1)}{(\alpha+\epsilon\nu)^7}. \tag{55c}$$

In this case the sign of BC is the sign of the quantity

$$-\epsilon[\alpha+\epsilon\nu+1][(\alpha+\epsilon\nu)(1+4\alpha)+3\alpha]. \tag{56}$$

Let us make $\epsilon=1$. For $\nu\in[0,\infty[$, we obtain u between $\sqrt{\alpha/(\alpha+1)}$ and 1, and we have $BC<0$, as is easy to see in Fig. 2.

If we take $\epsilon=-1$, we have, from (56), that $BC<0$ if ν belongs to

$$](1+\alpha)4\alpha/(1+4\alpha),1+\alpha[$$

and $BC>0$ otherwise. On the other hand, ν does not belong to $[\alpha,\alpha+1]$ since u must be real. Thus BC is in this case always positive and represented by a part of the straight line $b=1$, with

$$u\in]0,\sqrt{\alpha/(\alpha+1)}[\cup]1,+\infty[,$$

in Fig. 2.

V. THE DISPERSION RELATION: TRANSITION OF THE WAVE TRAIN BETWEEN FOCUSING AND DEFOCUSING STATES

In this last section, using the results of Sec. IV, and analyzing the dispersion relation [in the form (15) or (16)], we show the existence of points in the plane (k,ω), situated on the curve $\omega(k)$, where the transition between focusing and defocusing states in the propagation regime of the wave train occurs: by changing ω and k, so that the point representing the incident plane wave moves along the curve $\omega(k)$, the BC sign changes, when it passes through the mentioned points. This phenomenon is an electromagnetic analog of the Benjamin-Feir instability of the Stokes wave train in hydrodynamics.

Relation (16) allows us to study k as a function of ω and to plot the function $\omega(k)$ without giving its explicit expression, which is too complicated. We have, from (16) and for $\varphi\neq 0$, that $k_\pm=0$ for $\omega=0$, $k_-=0$ for $\omega=1+\alpha$, and $k_-\to\infty$ as $\omega\to\sqrt{\alpha(\alpha+\sin^2\varphi)}$. We have also the asymptotic expansion

$$k_\pm=\omega\pm\tfrac{1}{2}m_x-\tfrac{1}{8}\frac{m^2}{\omega}[4\alpha+1+(1-2\alpha)\sin^2\varphi]+O\left[\frac{1}{\omega^3}\right]. \tag{57}$$

For $\omega\to 0$, the slope of the tangents $\omega/k_\pm$ are

$$\frac{\omega}{k_-}\sim\left[\frac{\alpha}{\alpha+1}\right]^{1/2}<\frac{\omega}{k_+}\sim\left[\frac{\alpha+\sin^2\varphi}{\alpha+1}\right]^{1/2}. \tag{58}$$

In Fig. 3 we have represented the three branches of the dispersion relation $\omega(k)$. They are distinguished by the numbers 1, 2, and 3. Also we call $k_-=P$ ($k_+=N$) because k_- (k_+) represents a wave of positive (negative) helicity.

We arrive thus at the second main result of our work: to establish, for each branch of the dispersion relation,

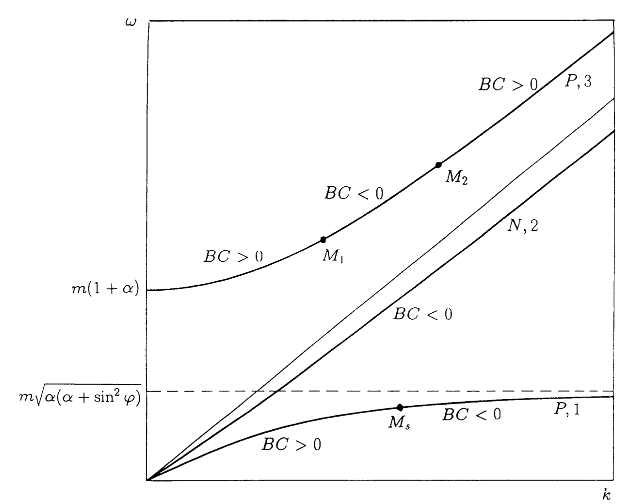

FIG. 3. Plot of ω against k with an indication of the points of transition of the regimes and the sign of BC in each region. Positive (negative) helicity is indicated P (N).

the points of transition between the focusing and defocusing regimes of the incident wave. Hence for each branch we have the following.

Branch 1. Here $u \in]0, \sqrt{\alpha/(1+\alpha)}[$ and we see (using Fig. 2) that there is one point of transition $M_s(k_s, \omega_s)$ in which the sign of BC changes, passing from $BC>0$ to $BC<0$ for $\varphi > \varphi_0$, where the angle φ_0 is determined using the point $(b_0, 0)$ in Fig. 2. Using the expression (50) of $b_0(\alpha)$ and the fact that

$$b = \alpha \cos^2\varphi / (\alpha + \sin^2\varphi) ,$$

for $u=0$, we obtain

$$\cos^2\varphi_0 = \frac{(1+\alpha) b_0(\alpha)}{\alpha + b_0(\alpha)} , \tag{59}$$

which determines φ_0. The values of k_s and ω_s (coordinates of the point of transition M_s) are determined as being the values of k and ω satisfying the functional relation

$$\mathcal{L}\left[b(\omega_s), \frac{\omega_s}{k_-(\omega_s)}\right] = 0 , \tag{60}$$

where

$$b(\omega_s) = \frac{\left\{1+\alpha\left[1-\left[\frac{k_-(\omega_s)}{\omega_s}\right]^2\right]\right\}^2 m^2\cos^2\varphi}{\left[1-\left[\frac{k_-(\omega_s)}{\omega}\right]^2\right]^2 \omega_s^2} . \tag{61}$$

This result shows that a right elliptically polarized wave, traveling through a saturated ferrite, may be focused or defocused depending on the position of the point representative of it in branch 1.

Branch 2. There are no points of transition in this branch. The values of u belong to

$$]\sqrt{(\alpha+\sin^2\varphi)/(1+\alpha)} , 1[$$

and we see from Fig. 2 that $BC<0$. This result shows that left elliptically polarized waves are stable. Note that the values of u belonging to

$$]\sqrt{\alpha/(1+\alpha)} , \sqrt{(\alpha+\sin^2\varphi)/(1+\alpha)}[$$

are never reached.

Branch 3. In this case u belongs to $[1, \infty[$ and from Fig. 2 we can see that for each $\varphi \neq 0, \pi/2$, there are always two points of transition, $M_1(k_1, \omega_1)$ and $M_2(k_2, \omega_2)$. In fact, all the curves defined in the (b,u) plane by $\varphi=$const pass through the point (1,1). Thus they all will cross the two curves $\mathcal{Q}(b,u)=0$ and $\mathcal{L}(b,u)=0$. The values of k_1, ω_1 and k_2, ω_2 are those for which

$$\mathcal{L}\left[b(\omega_1), \frac{\omega_1}{k_-(\omega_1)}\right] = 0 \tag{62}$$

and

$$\mathcal{Q}\left[b(\omega_2), \frac{\omega_2}{k_-(\omega_2)}\right] = 0 , \tag{63}$$

with $b(\omega)$ given by (61). On branch 3 and between M_1 and M_2, $BC<0$, and $BC>0$ elsewhere.

In the case of longitudinal propagation, we have $b=1$, and there are no transition points on each branch. The sign of BC is fixed for each branch. Hence on branch 2, $BC<0$ and on branches 1 and 3, $BC>0$. We have shown, thus, that right circularly polarized plane waves (negative helicity) or left circularly polarized ones (positive helicity) are modulationally stable or modulationally unstable, respectively. Finally, let us give the expressions for $\varphi(X,T)$ in (44) for this case.

On branch N in Fig. 3, an in-going (carrier) plane wave at $\xi \to -\infty$ is slowly modulated for its amplitude in the form of a tanh function. The expression of the corresponding dark soliton of (44), calculated using Hirota's method [10], and representing the defocusing case, is given by

$$\varphi(X,T) = \sqrt{\lambda} \exp i[KX - (K^2+2\rho^2)T] \frac{R^2}{2\lambda|E|}(1+ir) \times \tanh\left[\frac{RX + R^2(2K/R + r)T}{2} + ir\right] , \tag{64}$$

with

$$\rho = \left[\left|\frac{\lambda E}{2}\right|\right]^{1/2} , \quad r = \pm\left[\frac{2\lambda|E|}{R^2} - 1\right]^{1/2} \tag{65}$$

and with K, R, and λ being arbitrary constants, and R satisfying $|R| < 2\rho$.

On the two branches P in Fig. 3, focusing of the wave envelope occurs and some given initial data bunch into solitons of the form

$$\varphi(X,T) = \left[\frac{8}{E}\right]^{1/2} 2i\eta \frac{\exp[-2i\xi X - 4i(\xi^2-\eta^2)T]}{\cosh 2\eta(X - X_0 + 4\xi T)} , \tag{66}$$

where η, ξ, and X_0 are arbitrary constants. Using (64) or (66) in (42), and the result in (12a)–(12c), we can obtain the explicit form of solutions for $M^1_{1,s}$ and $H^1_{1,s}$ ($s=x,y,z$).

VI. EXPERIMENTAL PERSPECTIVES

Experiments made on yttrium iron garnet thin films [13] (which are not described by our model, because the hypothesis of isotropy is not verified in thin films, and the demagnetizing field is no more negligible) show that the difference between focusing and defocusing regimes can be observed experimentally. The pulsations of the focusing-defocusing transitions ω_s on branch 1 and ω_1 and ω_2 on branch 3, could thus be measured experimentally. These quantities are theoretically predicted by formulas (60), (62), and (63), which define them as implicit functions of the parameters α and φ of the external field. Numerically computed values of ω_1 and ω_2 could be compared with experimentally obtained ones.

VII. SUMMARY, CONCLUSIONS, AND PERSPECTIVES

We have studied the modulation of an electromagnetic wave in an infinite saturated ferromagnet in the presence of an external magnetic field. We have shown that this modulation is governed by the NLS equation. Envelope soliton solutions or dark-soliton solutions exist only if the coefficients of this NLS equation belong to a determined set of values in a given space of physical parameters. We have established these regions. We have analyzed the dispersion relation and we have shown that there are points of transition between focusing and defocusing for waves with positive helicity and no transition for negative helicity. The very important particular case of longitudinal propagation of plane waves was studied in detail and its explicit solution calculated. These facts determine for the first time a Benjamin-Feir-instability phenomenon in electromagnetic propagation in a saturated ferrite. All our results are valid under the hypothesis that the incident carrier wave is only one of the polarized waves allowed by the dispersion relation and if the two-wave interaction is not considered.

This work does not take into account the effect of transverse variation of the amplitude. The authors are presently working on this problem, introducing a second space variable in the perturbative calculus. Although this latter work is not achieved, we can announce here the main result: the $(2+1)$ dimensions generalization of Eq. (31) is not the so-called two-dimensional NLS equation obtained with replacing operator $\partial^2/\partial\xi^2$ in (31) by a two-dimensional Laplacian Δ, but an equation analog in form to Davey and Stewartson's [11], although nonintegrable in the general case.

In the longitudinal case (angle $\varphi=0$), one obtains the system

$$iA\frac{\partial g}{\partial\tau}+B\frac{\partial^2 g}{\partial\xi^2}+C\frac{\partial^2 g}{\partial\zeta^2}-Dg(|g|^2-\lambda)+Eg\Phi=0\ , \tag{67a}$$

$$\frac{\partial^2\Phi}{\partial\xi^2}=F\frac{\partial^2\Phi}{\partial\zeta^2}+G\frac{\partial^2|g|^2}{\partial\zeta^2}\ , \tag{67b}$$

where A, B, C, D, E, F, and G are real constants, ζ is a transverse space variable of the same order of magnitude as ξ, and Φ is an auxiliary field.

A, B, C, D, E, F, and G have been explicitly calculated: they never take the values for which system (67) is completely integrable and admits localized soliton solutions [12]. In the more general case where angle $\varphi\neq 0$, analog equations, but with two auxiliary fields, are obtained.

We are trying to find particular values of the parameters for which this system reduces to the integrable Davey-Stewartson equation; later, a study of properties of the system obtained in the general (nonintegrable) case should be done. We have left for future investigation the inclusion of dissipation in the model (Landau damping) which would lead to the nonlinear Ginsburg-Landau equation.

APPENDIX

There we give some mathematical details of the derivation of the equations of Sec. III.

Equations (10a) and (10b) give, using (8a) and (8b) and (9a) and (9b), in leading order $(0,n)$,

$$in\omega\mathbf{M}_0^n=\sum_{p+q=n}\mathbf{M}_0^p\times\mathbf{M}_0^q\ , \tag{A1}$$

$$n^2\omega^2(H_{0,s}^n+M_{0,s}^n)=n^2k^2H_{0,s}^n(1-\delta_{s,x})\ , \tag{A2}$$

where $s=x,y,z$.

This system has the particular solution

$$\mathbf{M}_0^n=\mathbf{m}\delta_{0,n}\ , \tag{A3}$$

$$\mathbf{H}_0^n=\alpha\mathbf{m}\delta_{0,n}\ , \tag{A4}$$

where α is a constant.

At order $(1,n)$ we obtain

$$in\omega\mathbf{M}_1^n=\mathbf{m}\times(\mathbf{H}_1^n-\alpha\mathbf{M}_1^n)\ , \tag{A5}$$

$$n^2\omega^2(H_{1,s}^n+M_{1,s}^n)=n^2k^2H_{1,s}^n(1-\delta_{s,x})\ . \tag{A6}$$

Equation (A6) gives the components $M_{1,s}^n$ as functions of $H_{1,s}^n$. Using this in (A5), we find a linear homogeneous system for $H_{1,x}^n$, $H_{1,y}^n$, and $H_{1,z}^n$. It reads

$$in\omega H_{1,x}^n+m_t\mu H_{1,z}^n=0\ , \tag{A7}$$

$$in\omega\gamma H_{1,y}^n-m_x\mu H_{1,z}^n=0\ , \tag{A8}$$

$$-(1+\alpha)m_tH_{1,x}^n+m_x\mu H_{1,y}^n+in\gamma\omega H_{1,z}^n=0\ . \tag{A9}$$

The determinant of this system, $\Delta(n)$, is

$$\Delta(n)=in\omega[-n^2\gamma^2\omega^2+\mu^2m_x^2+\gamma\mu(1+\alpha)m_t^2]\ , \tag{A10}$$

with γ,μ defined by (13) and (14).

For $n=1$, $\Delta(1)$ is zero if ω satisfies the dispersion relation

$$-\gamma^2\omega^2+\mu^2m_x^2+\gamma\mu(1+\alpha)m_t^2=0\ , \tag{A11}$$

which, when written in terms of α, k, ω, m_x, and m_t, gives Eq. (15).

Under this condition, the system has a nontrivial solution given by (12a), (12b), and (12c). Then for $n=2,3,\ldots$, $\Delta(n)\neq 0$, and we have the trivial solution (11). For $n=0$ $\Delta(0)=0$, and we can choose $\mathbf{M}_1^0=\mathbf{H}_1^0=\mathbf{0}$ [Eq. (19)]. This completes the solution at order $(1,n)$. At next order, we have the system

$$\begin{aligned}in\omega\mathbf{M}_2^n&-\mathbf{m}\times(\mathbf{H}_2^n-\alpha\mathbf{M}_2^n)\\&=(\mathbf{M}_1^1\times\mathbf{H}_1^1)\delta_{n2}+(\mathbf{M}_1^{1*}\times\mathbf{H}_1^1+\mathbf{M}_1^1\times\mathbf{H}_1^{1*})\delta_{n,0}\\&\quad+(\mathbf{M}_1^{1*}\times\mathbf{H}_1^{1*})\delta_{n,-2}-V\frac{\partial\mathbf{M}_1^n}{\partial\xi}\ ,\end{aligned} \tag{A12}$$

$$\begin{aligned}-\omega^2n^2(H_{2,s}^n&+M_{2,s}^n)+n^2k^2H_{2,s}^n(1-\delta_{1,x})\\&=-2in\omega V\frac{\partial}{\partial\xi}(H_{1,s}^n+M_{1,s}^n)\\&\quad+\left[2ink\frac{\partial}{\partial\xi}H_{1,s}^n\right](1-\delta_{1,x})\ .\end{aligned} \tag{A13}$$

For $|n|\geq 2$, (A13) gives

$$M_{2,x}^2=-H_{2,x}^2\ , \tag{A14}$$

$$M^2_{2,y} = -\gamma H^2_{2,y} \ , \tag{A15}$$

$$M^2_{2,z} = -\gamma H^2_{2,z} \ . \tag{A16}$$

Then (A12) can be reduced to a linear system with determinant

$$\Delta(n) = in\omega(1-n^2)\gamma^2\omega^2 \ . \tag{A17}$$

Thus, $\Delta(n)\neq 0$ if $|n|\geq 2$.

For $|n|>2$, the system is homogeneous and this shows (20). For $n=2$ the system is inhomogeneous. Solving it, we obtain (21a), (21b), and (21c). For $n=1$, using (A12) and (A13), we find an inhomogeneous linear system for the $\mathbf{H}^1_2$ components. It reads

$$i\omega H^1_{2,x} + \mu m_t H^1_{2,z} = -i\gamma\mu m_t V\frac{\partial g}{\partial \xi} + 2i\omega\gamma\alpha m_t \Omega\frac{\partial g}{\partial \xi} \ , \tag{A18}$$

$$i\gamma\omega H^1_{2,y} - \mu m_x H^1_{2,z} = i\gamma\mu m_x V\frac{\partial g}{\partial \xi} + 2i\omega m_x\Omega\frac{\partial g}{\partial \xi} \ , \tag{A19}$$

$$-(1+\alpha)m_t H^1_{2,x} + \mu m_x H^1_{2,y} + i\gamma\omega H^1_{2,z} = -\gamma^2\omega V\frac{\partial g}{\partial \xi} - 2\Omega(\gamma\omega^2 - \alpha\mu m_x^2)\frac{\partial g}{\partial \xi} \ . \tag{A20}$$

The determinant $\Delta(1)$ of the system is, in this case, zero due to the dispersion relation (15). Therefore, the system will have a solution only if the determinant of the augmented matrix is also zero. This condition is satisfied if [Eq. (24)]

$$V = V_g = \frac{\partial \omega}{\partial k} \ . \tag{A21}$$

Under this solvability condition we get (22a), (22b), (22c), (22d), and (22e). At the order (2,0), Eqs. (A12) and (A13) do not contain all the necessary information to determine completely $\mathbf{M}^0_2, \mathbf{H}^0_2$. We must go to the order (4,0) of Eq. (10b) to determine $\mathbf{M}^0_2$ as a function of $\mathbf{H}^0_2$. Using the results of the orders $(0,n)$ and $(1,n)$, we get

$$M^0_{2,s} = -H^0_{2,s}[\delta_{s,x} + \beta(1-\delta_{s,x})] \ , \tag{A22}$$

where β is defined by (26). Making use of these equations in (A12) and (A13), we find the components of $\mathbf{M}^0_2$ and $\mathbf{H}^0_2$ as in (25a), (25b), and (25c).

The next order, $(3,n)$, is given by the following set of equations:

$$\begin{aligned} n^2\omega^2(H^n_{3,s} + M^n_{3,s}) - n^2k^2 H^n_{3,s}(1-\delta_{s,x}) &= 2in\omega V\frac{\partial}{\partial \xi}(H^n_{2,s} + M^n_{2,s}) \\ &\quad - \left[V^2\frac{\partial^2}{\partial \xi^2} + 2in\omega\frac{\partial}{\partial \tau}\right](H^n_{1,s} + M^n_{1,s}) \\ &\quad - \left[2ink\frac{\partial}{\partial \xi}H^n_{2,s} + \frac{\partial^2}{\partial \xi^2}H^n_{1,s}\right](1-\delta_{s,x}) \ , \end{aligned} \tag{A23}$$

$$\begin{aligned} in\omega\mathbf{M}^n_3 - \mathbf{m}\times(\mathbf{H}^n_3 - \alpha\mathbf{M}^n_3) &= \sum_{p+q=n}(\mathbf{M}^p_1\times\mathbf{H}^q_2 + \mathbf{M}^p_2\times H^q_1) \\ &\quad - V\frac{\partial}{\partial \xi}\mathbf{M}^n_2 + \frac{\partial}{\partial \tau}\mathbf{M}^n_1 \ . \end{aligned} \tag{A24}$$

For the order (3,0), Eq. (A24) has a solution only if

$$\mathbf{m}\{2\,\mathrm{Re}[\mathbf{M}^2_1\times\mathbf{H}^{1*}_2 + \mathbf{M}^{1*}_2\times\mathbf{H}^1_1] - V\frac{\partial}{\partial \xi}\mathbf{M}^0_2\} = 0 \ . \tag{A25}$$

(Recall that $\mathbf{m}$ is defined by ($\mathbf{M}^0_0 = \mathbf{m}$). This equation determines $\Phi(\xi,\tau)$ in terms of $g(\xi,\tau)$ [Eq. (27)].

At order (3,1), Eq. (A23) gives the components of vector $\mathbf{M}^2_3$ as functions of the $\mathbf{H}^1_3$.

Using these expressions in Eq. (A24), we obtain a linear 3×3 system for the $\mathbf{H}^1_3$ components, whose determinant is $\Delta(1)=0$. Thus, this system has solutions only if the determinant of the augmented matrix is also zero. This last condition, which is very laborious to write explicitly, is the nonlinear Schrödinger equation (31).

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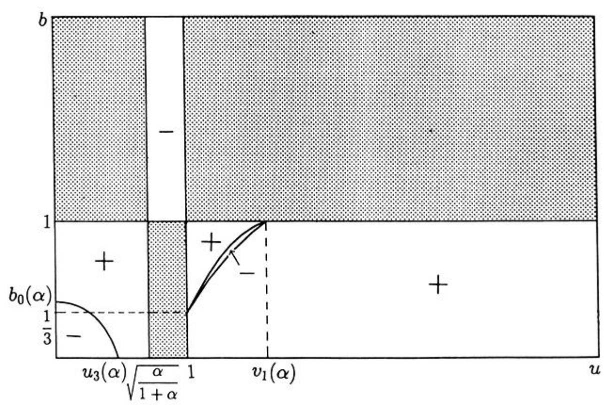

FIG. 2. The sign of BC in the (b,u) plane for α given ($\alpha=1$).